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Amenable Actions of Locally Compact Groups

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1. SUMMARY

We consider a locally compact group G with jointly continuous action $G \times Z \to Z$ on a locally compact space. The finite Radon (regular Borel) measures $M(G)$ act naturally on various function spaces supported on Z, including the continuous bounded functions $CB(Z)$. If Z supports a (not necessarily unique) nonnegative quasi-invariant measure ν we apply some recent studies [7] to define a Banach module action of $M(G)$ on $L^1(Z, \nu)$, and this induces a natural adjoint action on $L^\infty(Z, \nu) = L^1(Z, \nu)^*$. These actions of $M(G)$ (and of G) give us definitions of *amenability* of the action of G on various function spaces supported on Z, including $CB(Z)$ and $L^\infty(Z, \nu)$ (when there is a quasi-invariant ν on Z), corresponding to the existence of a left-invariant mean on these various spaces. When G acts on one of its coset spaces G/H, H a closed subgroup, there is always a quasi-invariant measure on G/H and the different definitions of amenability of the action $G \times G/H \to G/H$ all coincide. A number of manipulations of invariant means on groups (the case where $G = Z$) then carry over to the context of transformation groups. We apply them to explore the following problems.

Let $G \times G/H \to G/H$ be an amenable action of G on one of its coset spaces. Let ν be a quasi-invariant measure on G/H.

QUESTION. If $\epsilon > 0$ and compact set $K \subset G$ are given, is there a compact set $U \subset Z$ with $0 < \nu(U) < \infty$ and

$$\frac{\nu(xU \Delta U)}{\nu(U)} < \epsilon \qquad \text{all} \qquad x \in K?$$

Here $A \Delta B$ is the symmetric difference set; we are essentially trying to construct sets U sufficiently "large" that they are moved only

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slightly by the elements in prescribed compact set $K \subset G$. In Section 4 we shall see that this can be done if ν is *G-invariant*, generalizing the "Følner conditions" which have been shown to be valid in all amenable groups G, and that it can not be done if ν is only quasi-invariant.

In Section 5 these results are applied to determine some weak containment properties of induced representations on an amenable group G, the above considerations being applied to the coset spaces G/H for closed subgroups (G amenable will be seen to imply amenability of the action $G \times G/H \to G/H$). This allows us to answer a previously unresolved question for amenable groups (see [5]) by proving:

THEOREM. *If G is amenable, then for every closed subgroup $H \subset G$ and every irreducible unitary representation S of G we have S weakly contained in the representation $U^{S|H}$ induced from $S \mid H$, the restriction of S to H.*

2. Preliminaries on the Action of G on Function Spaces on Z

Hereafter $G \times Z \to Z$ will be a jointly continuous transformation group with G, Z locally compact Hausdorff spaces. If ν is a nonnegative quasi-invariant Radon measure on Z and m_G a fixed Haar measure for G (writing $dm_G(g) = dg$ for brevity) define $L^p(Z, \nu)$ $1 \leqslant p \leqslant \infty$ as the usual spaces of Borel functions (Borel sets being the σ-algebra generated by all open sets) identified when they coincide off a locally ν-null set in Z. Note that $L^\infty(Z, \nu)$ depends only on the equivalence class of ν (i.e., the family of null sets), while $L^1(Z, \nu)$ depends more explicitly on the nature of ν. We have $L^\infty(Z, \nu) = L^1(Z, \nu)^*$. Let $L(G)$, $L(Z)$ be the continuous functions with compact support, $C_0(G)$, $C_0(Z)$ the continuous functions vanishing at infinity, and $M(G) = C_0(G)^*$, $M(Z) = C_0(Z)^*$ the spaces of finite Radon measures with total variation norm.

For our purposes we are interested in actions of G and $M(G)$ on translation invariant function spaces X on Z. If $f \in CB(Z)$, the continuous bounded functions, we define the action of $M(G)$ directly via

$$\mu \,\square\, f(\xi) = \int f(g\xi)\, d\mu(g) \qquad \text{all} \qquad \xi \in Z.$$

Then $\mu \,\square\, f \in CB(Z)$ since $|\int f(g\xi)\, d\mu(g)| \leqslant \|\mu\| \cdot \|f\|_\infty$ all $\xi \in Z$, and $\xi_i \to \xi$ in $Z \Rightarrow f(g\xi_i) \to f(g\xi)$ uniformly on compacta in G, so $|\int f(g\xi_i)\, d\mu(g) - \int f(g\xi)\, d\mu(g)| \to 0$, as required. If Z supports a nonnegative quasi-invariant Radon measure ν (this will certainly be the case if G, Z are second countable, for we may use Haar measure on G to induce a finite quasi invariant measure on any one of the orbits in Z, these being σ-compact Borel sets), there is an obvious action of G on $L^\infty(Z, \nu)$, but there are technical difficulties in showing directly that our formula above gives a well-defined action of $M(G)$ on $L^\infty(Z, \nu)$. We apply the results of [7] which define Banach module actions $M(G) \times M(Z) \to M(Z)$ and $M(F) \times L^1(Z, \nu) \to L^1(Z, \nu)$; the desired action of $M(G)$ on $L^\infty(Z, \nu)$ will then appear as the adjoint of the action on $L^1(Z, \nu)$ since $L^\infty(\nu) = L^1(\nu)^*$.

We digress to summarize the necessary information from [7]. The action $M(G) \times M(Z) \to M(Z)$ is defined via[1]

$$\mu * \eta(\psi) = \iint_{G\times Z} \psi(g\xi)\, d\mu \times \eta(g, \xi), \quad \mu \in M(G),\ \eta \in M(Z),\ \psi \in C_0(Z).$$

Then we have (see [7])

$$\begin{aligned} (\mu_1 * \mu_2) * \eta &= \mu_1 * (\mu_2 * \eta), \\ \|\mu * \eta\| &\leqslant \|\mu\| \cdot \|\eta\|, \end{aligned} \tag{1}$$

and $M(Z)$ becomes a Banach module over $M(G)$.

If $g \in G$ and η is any Radon measure on Z, define $\eta_g(E) = \eta(g^{-1}E)$; for quasi-invariant ν we have $\nu_g \ll \nu$ for each $g \in G$ and there is a locally ν-integrable Radon–Nikodym derivative $(d\nu_g/d\nu)$ such that $\nu_g = (d\nu_g/d\nu) \cdot \nu$, which means:

$$\int f(g\xi)\, d\nu(\xi) = \int f(\xi)\, d[\nu_g](\xi) = \int f(\xi)(d\nu_g/d\nu)(\xi)\, d\nu(\xi) \quad \text{all} \quad f \in L(Z).$$

Now assume ν is a fixed nonnegative quasi-invariant measure on Z and embed $L^1(Z, \nu) \subset M(Z)$; then $L_1(Z, \nu)$ is a closed $M(G)$ submodule, for in Theorem 4.2 of [7] a theorem of P. Cohen (modified for Banach modules) is used to show that $L^1(G) * L^1(\nu) = L^1(\nu)$, so we have $(M(G) * L^1(G)) * L^1(\nu) = L^1(\nu)$ as required.

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[1] $(g, \xi) \to \psi(g\xi)$ is continuous on $G \times Z$, hence is $B(G \times Z)$ measurable, but unless G, Z are σ-compact, the product of Borel structures $B(G) \times B(Z)$ will be properly included in $B(G \times Z)$, and there will be some trouble defining this integral or applying Fubini's Theorem. However, the supports $\mathrm{supp}(\mu)$, $\mathrm{supp}\,(\eta)$ are σ-compact and one can readily verify that $\chi_{\mathrm{supp}(\mu)\times\mathrm{supp}(\eta)}(g, \xi)\psi(g\xi)$ is $B(G) \times B(Z)$ measurable—i.e., $\psi(g\xi)$ is $|\mu \times \eta|$-measurable, and this is sufficient for our purposes.

Now we resume consideration of the action of $M(G)$ on function spaces on Z. If $\mu \in M(G)$ and $f \in CB(Z)$, define

$$\mu \,\square\, f(\xi) = \int f(g\xi)\, d\mu(g), \tag{2}$$

and if Z supports a quasi-invariant measure ν, define the action of $\mu \in M(G)$ in $L^\infty(Z, \nu)$ via

$$\langle \mu \,\square\, f, \varphi\rangle = \langle f, \mu * \varphi\rangle, \qquad f \in L^\infty(Z, \nu), \quad \varphi \in L^1(Z, \nu). \tag{3}$$

When there is a quasi-invariant measure ν at hand the following lemma shows that formulae (2), (3) are compatible when we embed $CB(Z)$ in $L^\infty(Z, \nu)$ via the obvious norm decreasing (and possibly many to one) linear map. Let $L_c^\infty(Z)$ be the linear space of all bounded Borel measurable functions on Z (*not* identified modulo equivalence).

LEMMA 2.1. *If $f \in CB(Z)$, (2) determines a new function $\mu \,\square\, f \in CB(Z)$. Now let ν be a fixed nonnegative quasi-invariant measure on Z and $j_\nu : L_c^\infty(Z) \to L^\infty(Z, \nu)$ the canonical linear embedding which identifies functions differing only on a locally ν-null set. Then if $\mu \in M(G)$, $f \in L_c^\infty(Z)$ the function $\xi \to \int f(g\xi)\, d\mu(g)$ is bounded, everywhere defined, ν-measurable, and its equivalence class in $L^\infty(Z, \nu)$ is precisely the vector $\mu \,\square\, j_\nu(f)$ determined by (3), so the action of $M(G)$ commutes with the embedding $j_\nu : L_c^\infty(Z) \to L^\infty(Z, \nu)$.*

COROLLARY. *If $f \in L^\infty(Z, \nu)$ for quasi-invariant ν on Z, then $\delta_x \,\square\, f(\xi) = f(x\xi)$ locally ν—almost everywhere (a.e.) for each $x \in G$.*

Proof. We have done the first statement. Let ν and $\mu \in M(G)$ be fixed and let $f \in L(Z)$. Then (3) and the definition of $\mu * \varphi$ gives $\langle \mu \,\square\, (j_\nu f), \varphi\rangle = \langle f, \mu * \varphi\rangle = \int [\int f(g\xi)\, d\mu(g)]\, \varphi(\xi)\, d\nu(\xi) = \langle j_\nu(\mu \,\square\, f), \varphi\rangle$ for all $\varphi \in L^1(\nu)$. If $f \in CB(Z)$ and $\varphi \in L^1(\nu)$ take $\{f_n\} \subset L(Z)$ with $\|f_n\|_\infty \leqslant \|f\|_\infty$ and $f_n \to f$ pointwise on the σ-compact set $\operatorname{supp}(\mu) \cdot \operatorname{supp}(\varphi) \supset \operatorname{supp}(\mu * \varphi)$; then

$$\begin{aligned}
\langle \mu \,\square\, (j_\nu f), \varphi\rangle = \langle f, \mu * \varphi\rangle &\leftarrow \langle f_n, \mu * \varphi\rangle \\
&= \int \Big[\int f_n(g\xi)\, \varphi(\xi)\, d\nu(\xi)\Big]\, d\mu(g) \to \\
&\quad \int \Big[\int f(g\xi)\, \varphi(\xi)\, d\nu(\xi)\Big]\, d\mu(g) = \int \Big[\int f(g\xi)\, d\mu(g)\Big]\, \varphi(\xi)\, d\nu(\xi) \\
&= \langle j_\nu(\mu \,\square\, f), \varphi\rangle
\end{aligned}$$

since $\int f_n(g\xi)\,\varphi(\xi)\,d\nu(\xi) \to \int f(g\xi)\,\varphi(\xi)\,d\nu(\xi)$ for all $g \in G$. As noted in defining the action of $M(G)$ on $M(Z)$, $f(g\xi)$ is at least $|\mu \times \varphi\nu|$-measurable, so the use of Fubini's Theorem is justified. It is somewhat more difficult to show that if $f \in L_c^\infty(Z)$ and $\mu \in M(G)$, $\eta \in M(Z)$ then $f(g\xi)$ is at least $|\mu \times \eta|$-measurable on $G \times Z$, but this follows by simple modifications of 19.10, 19.11 in [8]. Thus if $\varphi \in L^1(\nu)$ is fixed, and we take $\{f_n\} \subset L(G)$ with $\|f_n\|_\infty \leqslant \|f\|_\infty$ and $f_n \to f$ ν—a.e. on $\text{supp}(\mu) \cdot \text{supp}(\varphi)$, the same argument shows

$$\langle \mu \,\square\, (j_\nu f), \varphi\rangle = \int \Big[\int f(g\xi)\,d\mu(g)\Big]\,\varphi(\xi)\,d\nu(\xi) = \langle j_\nu(\mu \,\square\, f), \varphi\rangle.$$

It is clear from Fubini's Theorem that the everywhere defined function $\mu \,\square\, f(\xi) = \int f(g\xi)\,d\mu(g)$ is bounded and ν-measurable.

Q.E.D.

Hereafter we will often not distinguish between $j_\nu(\mu \,\square\, f)$ and $\mu \,\square\, j_\nu f$ in discussing the action of $M(G)$ on $L^\infty(Z, \nu)$. One can readily verify the following properties.

$$\begin{aligned} &(\mu_1 * \mu_2) \,\square\, f = \mu_2 \,\square\, (\mu_1 \,\square\, f), \\ &\|\mu \,\square\, f\|_\infty \leqslant \|\mu\| \cdot \|f\|_\infty, \quad f \in CB(Z), \\ &\|\mu \,\square\, f\|_{\infty,\nu} \leqslant \|\mu\| \cdot \|f\|_{\infty,\nu}, \quad f \in L^\infty(Z, \nu) \text{ for quasi-invariant } \nu. \end{aligned} \tag{4}$$

There are several additional facts we shall need in discussing amenable action of G. Write $UCB_l(Z)$ for those $f \in CB(Z)$ such that $g \to \delta_g \,\square\, f$ is continuous from $G \to (CB, \|\cdot\|_\infty)$, the ***bounded left uniformly continuous functions***.[2] We note that $L(Z) \subset UCB_l(Z)$, hence $C_0(Z) \subset UCB_l(Z)$ since these are $\|\cdot\|_\infty$ limits of $L(Z)$. [If $\epsilon > 0$, U_0 a compact neighborhood of the unit $e \in G$, and $K = U_0^{-1}\,\text{supp}(f)$ for fixed $f \in L(Z)$, then $\xi \in K \Rightarrow$ There is a compact neighborhood $U(\xi)$ of e in G such that $|f(g\xi) - f(\xi)| < \epsilon$ all $g \in U(\xi)$; argument by contradiction shows there is also a neighborhood $N(\xi)$ of ξ such that $|f(g\xi') - f(\xi')| < \epsilon$ all $g \in U(\xi)$, $\xi' \in N(\xi)$. If we take a finite covering $\{N(\xi_1),..., N(\xi_k)\}$ of K and $U = U_0 \cap (\bigcap_{i=1}^k U(\xi_i))$, we readily compute $|f(g\xi) - f(\xi)| < \epsilon$ all $g \in U$, $\xi \in K$, which $\Rightarrow$ $\|\delta_g \,\square\, f - f\|_\infty < \epsilon$ if $g \in U$].

[2] In [6], where we consider the case $G = Z$ these are, unfortunately, referred to as the *right* uniformly continuous functions. Notice that our uniformity condition alone: $g_j \to g \Rightarrow \|\delta_{g_j} \,\square\, f - \delta_g \,\square\, f\|_\infty \to 0$, for a bounded function f, does not automatically give *continuity* of f; it says only that f is continuous on each orbit, but any function constant on orbits does this.

LEMMA 2.2. *Let ν be nonnegative quasi-invariant measure on Z and fix $x \in G$. For any $\varphi \in L^1(Z, \nu)$ we have*

$$\delta_x * \varphi(\xi) = \left(\frac{d\nu_x}{d\nu}\right)(\xi)\, \varphi(x^{-1}\xi), \qquad \nu\text{—a.e. on } Z.$$

Proof. If $f \in L(Z)$ then

$$\langle \delta_x * \varphi, f\rangle = \iint f(g\xi)\, d[\delta_x](g)\, \varphi(\xi)\, d\nu(\xi) = \int f(x\xi)\, \varphi(\xi)\, d\nu(\xi)$$
$$= \int f(\xi)\left(\frac{d(\nu_x)}{d\nu}(\xi)\right) \varphi(\xi)\, d\nu(\xi),$$

which $\Rightarrow$ $\delta_x * \varphi(\xi) = d(\nu_x)/d\nu(\xi)\, \varphi(\xi)$ locally ν—a.e. But it is shown in 12.2 of [8] that ν-*integrable* functions coincide locally ν—a.e. $\Rightarrow$ they coincide ν—a.e. Q.E.D.

LEMMA 2.3. *If $\{e_\alpha : \alpha \in J\}$ is an approximate identity for $L^1(G)$ with $\| e_\alpha \|_1 \to 1$, then $\| e_\alpha * \varphi - \varphi \|_{1,\nu} \to 0$ all $\varphi \in L^1(Z, \nu)$.*

Proof. Since we must have $\int e_\alpha(g)\, dg \to 1$ we may assume $\int e_\alpha dt = 1$ all α, and so for $\varphi \in L^1(\nu)$, $f \in L(Z)$:

$$\left|\langle e_\alpha * \varphi - \varphi, f\rangle\right| = \left|\iint f(g\xi)\, e_\alpha(g)\, \varphi(\xi)\, dg d\nu(\xi) - \iint e_\alpha(g) \varphi(\xi) f(\xi)\, dg d\nu(\xi)\right|$$
$$= \left|\int e_\alpha(g) \left[\int \varphi(\xi) f(g\xi) - \varphi(\xi) f(\xi)\, d\nu(\xi)\right] dg\right|$$
$$= \left|\int e_\alpha(g) \left[\int \left[\varphi(g^{-1}\xi)\, \frac{d\nu_g}{d\nu}(\xi) - \varphi(\xi)\right] f(\xi)\, d\nu(\xi)\right] dg\right|$$
$$= \left|\int e_\alpha(g) \left[\int (\delta_g * \varphi - \varphi(\xi))\, f(\xi)\, d\nu(\xi)\right] dg\right|$$
$$\leqslant \| f \|_\infty \cdot \int | e_\alpha(g)|\, \| \delta_g * \varphi - \varphi \|_{1,\nu}\, dg.$$

But $g \to \delta_g * \varphi$ is shown to be a continuous map $G \to (L^1(\nu), \| \cdot \|_{1,\nu})$ in [7], Theorem 4.18. As the mass of e_α is eventually gathered near the unit of G, we see that the right side goes to zero uniformly for all f with $\| f \|_\infty \leqslant 1$. Q.E.D.

LEMMA 2.5. *The spaces $C_0(Z) \subset UCB_l(Z) \subset CB(Z)$ are invariant under the action of $L^1(G)$ and $L^1(G) \square CB(Z) \subset UCB_l(Z)$. If ν is a quasi-invariant measure on Z and $f \in L_c^\infty(Z)$, $\varphi \in L^1(G)$ then $\varphi \square f(\xi) = \int f(g\xi)\, \varphi(g)\, dg$ is everywhere defined, bounded, ν-measurable, and has* *the property* $\| \delta_{g_i} \square (\varphi \square f) - \delta_g \square (\varphi \square f)\|_\infty \to 0$ *as* $g_i \to g$ *in* G, *but* $\varphi \square f$ *need not be continuous.*

Proof. We showed in Lemma 2.1 that if $f \in L_c^\infty(Z)$ and ν quasi-invariant, then $\mu \square f$ is ν-measurable and if $j_\nu : L_c^\infty \to L^\infty(\nu)$ is the canonical map associated with ν we have $j_\nu(\mu \square f) = \mu \square j_\nu f$ for $\mu \in M(G)$. Let $f \in L_c^\infty$, fix $\xi \in Z$, and let $g_i \to g$ in G. Then if $\varphi \in L^1(G)$ and we take $d\mu = \varphi dt$, we have $\varphi \square f$ defined everywhere and

$$\begin{aligned}
&|\varphi \square f(g_j\xi) - \varphi \square f(g\xi)| \\
&= \left|\int f(xg_j\xi)\, \varphi(x)\, dx - \int f(xg\xi)\, \varphi(x)\, dx\right| \\
&= \left| \Delta_G(gg_j^{-1}) \int f(xg\xi)\, \varphi(xgg_j^{-1})\, dx - \int f(xg\xi)\, \varphi(x)\, dx \right| \\
&\leqslant \|f(xg\,\cdot\,)\|_\infty \cdot \| \Delta_G(gg_j^{-1})\, \varphi(\,\cdot\, gg_j^{-1}) - \varphi(\,\cdot\,)\|_{1,G}\,.
\end{aligned}$$

As is well known, the right-hand norm converges to zero as $g_j \to g$ in G, and this convergence is independent of $\xi \in Z$. If G is the real line acting irrationally on Z the torus, consider f to be 0 on the orbit of the unit in Z and 1 elsewhere; then $\varphi \square f = f$ and is not continuous. We have noted that $\varphi \square f$ is continuous on Z if $f \in CB(Z)$, which shows the invariance of $UCB_l(Z)$ and $CB(Z)$ under $L^1(G)$. For $L^1(G) \square C_0(Z) \subset C_0(Z)$ it suffices to show $L^1(G) \square L(Z) \subset C_0(Z)$, which is left to the reader. Q.E.D.

3. Amenable Action of a Transformation Group

We adopt the terminology of [*6*] and say that a locally compact group G is *amenable* if there is a left invariant mean (LIM) on $CB(G)$—see [*6*], Section 3.2, where it is shown that we get equivalent definitions by considering the action of G in $L^\infty(G)$, $CB(G)$, $UCB_l(G)$, and $UCB(G)$.

If $G \times Z \to Z$ is a jointly continuous transformation group and if X is one of the spaces $CB(Z)$, $UCB_l(Z)$ [or $L^\infty(Z, \nu)$ if we assume Z supports a nontrivial quasi-invariant measure ν] we define a *mean* m on X to be any linear functional on X such that $m(\bar{f}) = m(f)^-$, $f \geqslant 0 \Rightarrow m(f) \geqslant 0$ [$f \geqslant 0$ loc. ν—a.e. $\Rightarrow m(f) \geqslant 0$] and $m(1) = 1$. Evidently $\| m \| = 1$ in X^* if X is equipped with the sup norm $\| \cdot \|_\infty$ [local ess. sup. norm $\| \cdot \|_{\infty,\nu}$ if $X = L^\infty(Z, \nu)$] and the set $\Sigma(X)$ of all means of X is a $\sigma(X^*, X)$-compact convex set. A mean m is *left invariant* (m a LIM) if $m(\delta_g \square f) = m(f)$ all $g \in G, f \in X$ and we say G

has *amenable action on* X if such a LIM exists. A mean m on X is *topologically left invariant* (m a topological LIM) if $m(\varphi \square f) = m(f)$ all $\varphi \in P(G)$, $f \in X$ where $P(G) = \{\varphi \in L^1(G) : \varphi \geqslant 0, \|\varphi\|_1 = 1\}$; this is a proper definition since the spaces $CB(Z)$, $UCB_l(Z)$, and $L^\infty(Z, \nu)$ [assuming Z supports some quasi-invariant ν] are invariant under the action of $L^1(G)$ by Lemma 2.4 and action of $\varphi \in P(G)$ preserves order and constant functions. The measures

$$\{\mu \in M(Z) : \mu \geqslant 0, \|\mu\| = 1\}$$

give means on $X = CB(Z)$, $UCB_l(Z)$ and form a $\sigma(X^*, X)$-dense convex subset of $\Sigma(X) \subset X^*$, as is seen by trivially modifying the discussion in [6], Section 1.1. If Z supports a nontrivial quasi-invariant ν then the measures corresponding to $P(\nu)$ are $\sigma(X^*, X)$ dense in $\Sigma(X)$ for $X = L^\infty(\nu)$, $CB(Z)$, $UCB_l(Z)$.

We now show that in the simple case where G acts on one of its coset spaces $G/H = \{xH : x \in G\}$, where H is a closed subgroup, the notion of amenable action $G \times G/H \to G/H$ does not depend on the function space X. This is a pleasant generalization of the "main equivalence theorem" Theorem 2.2.1 in [6]; for more general actions $G \times Z \to Z$ the situation does not seem to be pleasant.

THEOREM 3.1. *Let G have jointly continuous action $G \times Z \to Z$. The four following statements are equivalent, and we say G acts amenably on Z if any one of them holds.*

(1) [(1A)] *There exists a [topological]* LIM *on* $UCB_l(Z)$.

(2) [(2A)] *There exists a [topological]* LIM *on* $CB(Z)$.

Moreover, (*G amenable*) $\Rightarrow$ (2A). *Now assume Z admits at least one nonzero, nonnegative quasi-invariant measure ν and consider the (nonvacuous) statements*:

(3) [(3A)] *For some nontrivial quasi-invariant ν, there exists a [topological]* LIM *on* $L^\infty(Z, \nu)$

(4) [(4A)] *For all nontrivial quasi-invariant ν, there exists a [topological]* LIM *on* $L^\infty(Z, \nu)$.

In this context we have implications

$$\begin{tikzcd}[column sep=small] & & (4) \arrow[dr] & & \\ (G\ \textit{amenable}) \Rightarrow & (4A) \arrow[ur] \arrow[dr] & & (3) & \Rightarrow (G\ \textit{acts amenably on}\ Z); \\ & & (3A) \arrow[ur] & & \end{tikzcd}$$

furthermore; (4A) $\not\Rightarrow$ (*G amenable*) *and* (3A) $\not\Rightarrow$ (4A), (3) $\not\Rightarrow$ (4).

Proof. We first note that every topological LIM on one of our function spaces X is necessarily a LIM, proving $(nA) \Rightarrow (n)$: in fact if we fix $\varphi \in P(G)$ we have $m(\delta_x \,\square\, f) = m(\varphi \,\square\, (\delta_x \,\square\, f)) = m((\delta_x * \varphi) \,\square\, f) = m(f)$. Moreover (4) $\Rightarrow$ (3), (4A) $\Rightarrow$ (3A) is trivial: (2) $\Rightarrow$ (1) and (2A) $\Rightarrow$ (1A) follow by restricting means on $CB(Z)$ to $UCB_l(Z)$; if ν is any quasi-invariant measure on Z, the canonical map $j_\nu : (CB(Z), \|\cdot\|_\infty) \to (L^\infty(\nu), \|\cdot\|_{\infty,\nu})$ is linear, norm-decreasing, order-preserving ($f \geqslant 0 \Rightarrow j_\nu f \geqslant 0$ in $L^\infty(\nu)$), carries constant functions to constant functions, and commutes with the action of G or $L^1(G)$. Thus any [topological] LIM on $L^\infty(\nu)$ gives a [topological] LIM on $CB(Z)$ when composed with j_ν and we see that (3) $\Rightarrow$ (2), (3A) $\Rightarrow$ (2A).

Conversely, any LIM m on $UCB_l(Z)$ is itself topologically left invariant, so (1) $\Rightarrow$ (1A). It suffices to show $m(\varphi \,\square\, f) = m(f)$ all $f \in UCB_l(Z)$, all $\varphi \in L(G) \cap P(G)$ (so φ has compact support), since these are $\|\cdot\|_1$-dense in $P(G)$. Because $g \to \delta_g \,\square\, f$ is $\|\cdot\|_\infty$-continuous, the Bochner integral $\int \varphi(g)\, \delta_g \,\square\, f\, dg$ is a well defined element in $UCB_l(Z)$; we assert that $\varphi \,\square\, f = \int \varphi(g)\, \delta_g \,\square\, f\, dg$. But once this is established it follows that for m, or any other linear functional $m \in (UCB_l)^*$, we have

$$m(\varphi \,\square\, f) = \int \varphi(g)\langle m, \delta_g \,\square\, f\rangle\, dg$$

$$= m(f) \int \varphi(g)\, dg = m(f).$$

To verify the assertion about $\varphi \,\square\, f$, just consider the point evaluation functionals $m_\xi : f \to f(\xi)$ which are sufficiently numerous to distinguish vectors in UCB_l; we have

$$m_\xi\left(\int \varphi(g)\, \delta_g \,\square\, f\, dg\right) = \int \varphi(g) f(g\xi)\, dg = \varphi \,\square\, f(\xi).$$

Next we prove (1A) $\Rightarrow$ (2A) by lifting a given topological LIM m on $UCB_l(Z)$ to one on $CB(Z)$. Let $E = E^{-1} \subset G$ be any compact neighborhood of the unit in G, $\varphi_E \in P(G)$ its normalized characteristic function. Define $\bar{m}(f) = m(\varphi_E \,\square\, f)$ for $f \in CB(Z)$; we have seen that $L^1(G) \,\square\, CB(Z) \subset UCB_l(Z)$. It is a straightforward matter to see that $f \to \varphi_E \,\square\, f$ is an order preserving map from $CB(Z)$ into $UCB_l(Z)$ which preserves constant functions, thus $\bar{m}$ is a well defined ***mean*** on $CB(Z)$. To see $\bar{m}$ is a topological LIM we must show this map commutes with the action of $L^1(G)$. Let $\{e_\alpha\} \subset P(G)$ be an approximate identity for $L^1(G)$; then for $\varphi \in P(G)$, $f \in CB(Z)$ we have:

$$\begin{aligned}\bar{m}(\varphi \square f) &= m(\varphi_E \square (\varphi \square f)) = m((\varphi * \varphi_E) \square f) \leftarrow m((e_\alpha * \varphi * \varphi_E) \square f)\\ &= m((\varphi * \varphi_E) \square (e_\alpha \square f)) = m(e_\alpha \square f)\\ &= m((e_\alpha * \varphi_E) \square f) \rightarrow m(\varphi_E \square f) = \bar{m}(f),\end{aligned}$$

as required.

Note. We have constructed an *equivariant* [commutes with actions of $L^1(G)$] "smoothing operator" from $CB(Z) \rightarrow UCB_l(Z)$ and this allows us to lift any [topological] LIM in the opposite direction proving (1A) $\Rightarrow$ (2A) and the first part of our Theorem. If Z supports a quasi-invariant ν, we could similarly prove (1A) $\Rightarrow$ (4A), thereby establishing that (1) $\cdots$ (4), (1A) $\cdots$ (4A) are all equivalent, *if* we could devise an equivariant smoothing operator $L^\infty(\nu) \rightarrow UCB_l(Z)$. Example 1 below shows (3A) $\not\Rightarrow$ (4A), so this can not be done in general; notice that the process above smoothes functions on each G orbit separately and does not even give a continuous function if we apply it to a function $f \in L_c^\infty(Z)$. We shall see that if G has *transitive* action it is usually possible to carry through this program; if ν lives on a dense orbit this is not enough, as one can see by contemplating the irrational action of the real line on the torus.

The following examples provide the counterexamples required for our Theorem.

EXAMPLE 1. G = free group on two-generators (non-amenable). $Z = G \cup \{\infty\}$, both with discrete topology. Let ν_1 = point mass on $\{\infty\}$, ν_2 = Haar (counting) measure on the subset $G \subset Z$; both are quasi-invariant if we let G act as left translation on $G \subset Z$ and leave fixed the point $\{\infty\} \subset Z$. Now $L^\infty(Z, \nu_2) \cong CB(G)$ via an isomorphism which commutes with the action of G, thus there is no LIM on $L^\infty(Z, \nu_2)$. There is obviously a LIM on $L^\infty(Z, \nu_1) \cong \mathbf{C}$.

EXAMPLE 2. Let $G = SO(3, \mathbf{R})$ with the *discrete* topology. Then G contains a copy of the free group on two generators and is therefore nonamenable [(see [*6*]), Example 1.2.8]. Let $Z = S^2$ (the unit sphere in $\mathbf{R}^3$) and let G have the obvious action as proper rotations of Z. Let G' be $SO(3, \mathbf{R})$ with its usual topology, fix a point $\xi \in Z$, and let $H = \{x \in G : x(\xi) = \xi\}$; H is a closed subgroup on G' and the (jointly continuous) action $G' \times Z \rightarrow Z$ is equivalent to the action $G' \times G'/H \rightarrow G'/H$. As is well known the homogeneous space $G'/H = Z$ admits an essentially unique quasi-invariant measure ν (ν' and ν'' quasi-invariant $\Rightarrow \nu' \sim \nu''$ in the sense that $\nu' \ll \nu''$ and $\nu'' \ll \nu'$, so they have the same class of null sets), see [*1*], Section 2, No. 5. Thus Z admits ν as the essentially unique quasi-invariant measure under the action $G \times Z \to Z$. There is obviously a (topological) LIM on $L^\infty(Z, \nu)$. In fact there is a unique finite normalized G-invariant measure on Z. Thus (4A) $\not\Rightarrow$ (amenable).

It is obvious (G amenable) $\Rightarrow$ (2A), for if we fix $\xi_0 \in Z$ the map $\pi : g \to g(\xi_0)$ is continuous and induces a map $CB(Z) \to CB(G)$ which is norm-decreasing, order-preserving, preserves constants, and commutes with the action of $L^1(G)$. Topological left-invariant means exist on $CB(G)$ and are carried to topological LIM on $CB(Z)$ which "live" on the orbit closure $G(\xi_0)^-$. Now assume ν is a quasi-invariant measure on Z and show (G amenable) $\Rightarrow$ (4A). Fix $\xi_0 \in \operatorname{supp}(\nu)$. If U is any compact neighborhood of ξ_0 in Z then $\infty > \nu(U) > 0$ and we may define the normalized characteristic function φ_U. Define the linear map $A : L_c^\infty(Z) \to CB(G)$ via

$$Af(g) = \int f(g\xi)\, \varphi_U(\xi)\, d\nu(\xi);$$

we have $Af \in CB(G)$ since $g_j \to g$ in $G \Rightarrow$

$$\begin{aligned}
&| Af(g_j) - Af(g)| \\
&\quad = \left|\int f(g_j\xi)\, \varphi_U(\xi)\, d\nu(\xi) - \int f(g\xi)\, \varphi_U(\xi)\, d\nu(\xi)\right| \\
&\quad \leqslant \int |f(\xi)| \left| \varphi_U(g_j^{-1}\xi) \frac{d(\nu_{g_j-1})}{d\nu}(\xi) - \varphi_U(g^{-1}\xi) \frac{d(\nu_g)}{d\nu}(\xi)\right| d\nu(\xi) \\
&\quad \leqslant \|f\|_\infty \cdot \| \delta_{g_j} * \varphi_U - \delta_g * \varphi_U \|_{1,\nu} .
\end{aligned}$$

The right-hand term goes to zero by 4.18 of [7]; recall that $\delta_x * \varphi(\xi) = \varphi(x^{-1}\xi)\, d(\nu_x)/d\nu(\xi)$ for $x \in G$, $\varphi \in L^1(Z, \nu)$. If $f = 0$ loc. ν—a.e., then $f(g\,\cdot) = 0$ loc. ν—a.e. too and since $\varphi_U \in L^1(\nu)$, $Af(g) = 0$ all $g \in G$. Moreover $f \geqslant 0 \Rightarrow Af \geqslant 0$, and $A(1) = 1$. Thus A factors through $j_\nu : L_c^\infty(Z) \to L^\infty(Z, \nu)$ to give a linear, order-preserving map $\tilde{A} : L^\infty(Z, \nu) \to CB(G)$. If $\psi \in L^1(G)$ we have[3]

$$\begin{aligned}
\psi \mathbin{\square} Af(g) &= \int \psi(x)\, Af(xg)\, dx \\
&= \iint \psi(x)\, f(xg\xi)\, \varphi_U(\xi)\, dx d\nu(\xi) \\
&= \int \varphi_U(\xi) \left[\int \psi(x)\, f(xg\xi)\, dx\right] d\nu(\xi) = A(\psi \mathbin{\square} f)
\end{aligned}$$

[3] We define action $(\psi, f) \to \psi \mathbin{\square} f$ rather than the usual action $(\psi, f) \to \psi * f$ of $L^1(G)$ on $CB(G)$, in order to have the same sort of action in the system $G \times G \to G$ as we have defined for a transformation group $G \times Z \to Z$. The notation for transformations groups is simpler this way, and we have the elementary relation $\psi \mathbin{\square} f = \psi^\sim * f$ where $\psi^\sim(x) = \psi(x^{-1})\Delta(x^{-1})$. Note that $\psi \to \psi^\sim$ is an isometric anti-isomorphism of $L^1(G)$.

so $\tilde{A}$ commutes with the actions of $L^1(G)$ in $L^\infty(Z, \nu)$ and $CB(G)$. Thus a (topological) LIM on $CB(G)$ lifts back via $\tilde{A}$ to a similar mean on $L^\infty(\nu)$. Q.E.D.

COROLLARY 3.2. *A locally compact group G is amenable $\Leftrightarrow$ G acts amenably on every coset space G/H where H is a closed subgroup.*

Proof. For ($\Leftarrow$) consider $H = \{e\}$; we have seen G amenable $\Rightarrow$ there is a LIM on $CB(G/H)$ in the first part of Theorem 3.1. Q.E.D.

We note that every coset space G/H admits a non-trivial quasi-invariant measure ν which is essentially unique (two such measures are mutually absolutely continuous), as in [*1*], Section 2, No. 5; the canonical map $G \to G/H$ is continuous and open. Now if G acts transitively on space Z fix any $\xi_0 \in Z$, let $H_0 = \{x \in G : x(\xi_0) = \xi_0\}$; then $xH_0 \to x(\xi_0)$ is an equivariant continuous bijection $G/H_0 \to Z$ but need not be open (see Example 2 above). If Z is not homeomorphic to G/H_0 it is not clear that Z supports any quasi-invariant measures (nor is uniqueness clear); if G/H_0 is σ-compact the map is open by Category argument. One might try to generalize the following theorem to transitive transformation groups $G \times Z \to Z$ where Z is assumed to support a nontrivial quasi-invariant measure ν (taking this ν is the statement of the theorem); however if $G/H_0 \to Z$ is not bicontinuous, the map $f \to \varphi \,\square\, f$ does not carry $L_c^\infty(Z)$ into $CB(Z)$, see Example 2 again, and the proof we give breaks down.

THEOREM 3.3. *Let H be a closed subgroup of G, and take the usual action $G \times G/H \to G/H$. In this situation all the statements* (1) $\cdots$ (4), (1A) $\cdots$ (4A) *are equivalent in Theorem* 3.1.

Proof. It suffices to prove (2A) $\Rightarrow$ (4A). Let $\varphi \in P(G)$ be fixed. Then if $f \in L_c^\infty(Z)$, define $\varphi \,\square\, f(\xi) = \int f(g\xi)\,\varphi(g)\,dg$ as usual. We have seen in Lemma 2.5 that $g_i \to g$ in $G \Rightarrow |\varphi \,\square\, f(g_i\xi) - \varphi \,\square\, f(g\xi)| \to 0$ uniformly in $\xi \in Z = G/H$. Fix $\xi_0 = g_0 H \in Z$ and define the continuous, open, equivariant surjection $G \to G/H = Z$ via $x \to x(\xi_0) = xg_0H$. If $x \in G$ then any base of neighborhoods of x in G maps to a base for the neighborhoods of $\xi = x(\xi_0)$ in G/H; it follows immediately that $\varphi \,\square\, f$ is *continuous* on Z. As in Lemma 2.5 it is clear that the map $f \to \varphi \,\square\, f$ of $L_c^\infty(Z) \to CB(Z)$ factors through the canonical map $L_c^\infty(Z) \to L^\infty(Z, \nu)$ for any quasi-invariant ν on Z to give a linear, order preserving, equivariant (under action of G and $L^1(G)$) map of

$L^\infty(Z, \nu) \to CB(Z)$ which preserves constants. Thus any topological LIM on $CB(Z)$ lifts to one on $L^\infty(Z, \nu)$. Q.E.D.

We shall apply amenability of the action $G \times G/H \to G/H$, which follows from Corollary 3.2 when G is amenable, to answer questions about induced representations for amenable G. To do this we need to consider $L^\infty(G/H)$, defined with respect to the essentially unique quasi-invariant measure on G/H, and the analog of Reiter's condition (see [6], Section 3.2) for amenable transformation groups. This in turn depends on the generalized notions of weak (strong) convergence to left invariance.

Let ν be a fixed quasi-invariant measure on $Z = G/H$, define $P(\nu) = \{\varphi \in L^1(\nu) : \varphi \geqslant 0 \ \ \nu\text{—a.e.}, \ \|\varphi\|_{1,\nu} = 1\}$. A net $\{\varphi_\alpha\} \subset P(\nu)$ *converges weakly* (*strongly*) *to left invariance* if $\delta_x * \varphi_\alpha - \varphi_\alpha \to 0$ for all $x \in G$, in the $\sigma(L^\infty(\nu), L^1(\nu))$-topology ($\|\cdot\|_{1,\nu}$-topology); note that $\|\varphi\|_{1,\nu} =$ norm as a functional in $L^\infty(\nu)^* \supset L^1(\nu)$; there are similar notions of *convergence to topological left invariance* in which $\varphi * \varphi_\alpha - \varphi_\alpha \to 0$ all $\varphi \in P(G)$, in these topologies. It is easily seen that convergence to topological invariance $\Rightarrow$ convergence to left invariance for a net in $P(\nu)$. We leave the reader to check that we can prove the following lemma exactly as in [6], Section 2.4.

LEMMA 3.4. *Let G have jointly continuous action on Z, and assume Z supports a nonnegative quasi-invariant measure ν. There is a net in $P(\nu)$ which is weakly convergent to (topological) left invariance $\Leftrightarrow$ there is a net strongly convergent to (topological) left invariance. Every $\sigma(L^\infty(\nu)^*, L^\infty(\nu))$-limit point of a net in $P(\nu) \subset (L^\infty(\nu))^*$ which converges weakly to (topological) left invariance is a (topological)* LIM *on $L^\infty(Z, \nu)$.*

In this context, if $L^\infty(Z, \nu)$ admits a (topological) LIM m we can produce nets in $P(\nu)$ which are weakly convergent to (topological) left invariance by noticing that $P(\nu) \subset (L^\infty(\nu))^*$ is $\sigma(L^\infty(\nu)^*, L^\infty(\nu))$-dense in the set of all means on $L^\infty(\nu)$, and taking any net $\{\varphi_\alpha\} \subset P(\nu)$ such that $\varphi_\alpha \to m$ in this topology. If ν is a quasi-invariant measure on a coset space G/H and if $G \times G/H \to G/H$ is an *amenable* action, it follows from 3.3 and 3.4 that there is a net in $P(\nu)$ *strongly* convergent to topological left invariance.

Trivial alterations of the proof of Theorem 3.2.1 in [6] gives the following generalization of Reiters condition (P_1); we omit details.

THEOREM 3.5. *Let G have jointly continuous action on Z; assume that Z supports a quasi-invariant measure ν and assume there exists a net in $P(\nu)$ strongly convergent to topological (hence also to simple) left* *invariance. If $\epsilon > 0$ and compact set $K \subset G$ are given, there exists a $\varphi \in P(\nu)$ such that*

$$\| \delta_x * \varphi - \varphi \|_{1,\nu} = \int \left| \varphi(x^{-1}\xi) \frac{d(\nu_x)}{d\nu} (\xi) - \varphi(\xi) \right| d\nu(\xi) < \epsilon$$

for all $x \in K$.

Thus if (ϵ, K) are given we can find a $\varphi \in P(\nu)$ which is not moved very much in $(L^1(\nu), \| \cdot \|_{1,\nu})$ by any translation $x \in K$.

COROLLARY. *Let G act amenably on the coset space G/H, H a closed subgroup in G. Let ν be any quasi-invariant measure on G/H. Then if $\epsilon > 0$ and compact set $K \subset G$ are given, there exists a $\varphi \in P(\nu)$ such that $\| \delta_x * \varphi - \varphi \|_{1,\nu} < \epsilon$ all $x \in K$.*

4. APPLICATION: CONSTRUCTING LARGE SETS IN Z

It is known [2] that, if G is amenable, then for any compact set $K \subset G$, it is possible to construct a compact set U which is "large" with respect to left translation by elements $x \in K$; i.e. the following "Følner condition" holds.

(FC) For any $\epsilon > 0$ and compact set $K \subset G$; there exists a compact set $U \subset G$ such that $0 < | U | < \infty$

$$\frac{| xU \Delta U |}{| U |} < \epsilon \qquad \text{all} \quad x \in K,$$

where $| E |$ is Haar measure in G.

We can generalize this when Z supports an *invariant* measure ν such that $L^\infty(Z, \nu)$ admits a topological LIM. Recall that if Z is a coset space $Z = G/H$, then G/H supports quasi-invariant measures, but supports an invariant measure $\Leftrightarrow \Delta_G \mid H = \Delta_H$ where Δ_G, Δ_H are the modular functions; we have seen in Theorem 3.4 that, if ν is quasi-invariant, there is a topological LIM on $L^\infty(G/H, \nu) \Leftrightarrow G$ acts amenably on G/H.

THEOREM 4.1. *Let G have jointly continuous action on Z and assume Z supports an invariant measure ν such that $L^\infty(Z, \nu)$ admits a topological LIM. Then if $\epsilon > 0$, compact set $K \subset G$, are given there exists a compact set $U \subset Z$ such that $0 < \nu(U) < \infty$ and $\nu(xU\Delta U)/\nu(U) < \epsilon$ all $x \in K$.*

Proof. The proof is similar to that in the known case where $(Z, \nu) = (G, m_G)$, which is presented in [6], Section 3.6; for completeness we sketch the revised proof.

First we show that the following condition holds (generalizing Lemma 3.6.2 in [6]):

(FC*) Given $\epsilon > 0$, $\delta > 0$ and compact $K \subset G$ there exist Borel sets $U \subset Z$ and $N \subset K$ such that $0 < \nu(U) < \infty$, $|N| < \delta$ and $\nu(xU \Delta U)/\nu(U) < \epsilon$ all $x \in K \sim N$.

We may assume $|K| > 0$ (otherwise take $N = K$); by Theorem 3.5 there is a $\varphi \in P(Z, \nu)$ such that $\| \delta_x * \varphi - \varphi \|_{1,\nu} < \epsilon\delta / |K|$ all $x \in K$. We may replace φ with a function of the form $\varphi = \sum_{i=1}^{N} \lambda_i \varphi_{A_i}$ where $\varphi_A(\xi)$ is the normalized characteristic function of $A \subset Z$ ($\varphi_A = [1/\nu(A)]\, \chi_A$ so $\varphi_A \in P(Z, \nu)$), $\lambda_i > 0$ with $\sum_{i=1}^{N} \lambda_i = 1$, and $A_1 \supset \cdots \supset A_N$ are compacta with $\infty > \nu(A_i) > 0$. Then since $A_1 \supset \cdots \supset A_N$,

$$(*) \qquad \begin{aligned} \frac{\epsilon\delta}{|K|} > \| \delta_x * \varphi - \varphi \|_{1,\nu} &= \sum_{i=1}^{N} \lambda_i \| \delta_x * \varphi_{A_i} - \varphi_{A_i} \|_{1,\nu} \\ &= \sum_{i=1}^{N} \lambda_i \frac{\nu(xA_i \Delta A_i)}{\nu(A_i)} \end{aligned}$$

[since ν is G-invariant, $\delta_x * \varphi_A(\xi) = \varphi_A(x^{-1}\xi)(d(\nu_x)/d\nu)(\xi) = \varphi_A(x^{-1}\xi)$, and $\| \delta_x * \varphi_A - \varphi_A \|_{1,\nu} = \nu(xA\Delta A)/\nu(A)$]. Now $x \to \delta_x * \varphi$ is continuous from $G \to L^1(Z, \nu)$ by Theorem 4.18a [7]; integrating $(*)$ over K we get

$$\epsilon\delta > \int_K \| \delta_x * \varphi - \varphi \|_{1,\nu}\, dx = \sum_{i=1}^{N} \lambda_i \int_K \frac{\nu(xA_i \Delta A_i)}{\nu(A_i)}\, dx$$

so there is an $1 \leqslant i \leqslant N$ such that

$$\int_K \frac{\nu(xA_i \Delta A_i)}{\nu(A_i)}\, dx < \epsilon\delta;$$

thus the integrand cannot be $\geqslant \epsilon$ throughout any set $N \subset K$ with $|N| \geqslant \delta$, and $A = A_i$ works. Now a straightforward paraphrase of the proof of Lemma 3.6.4 in [6] proves (FC*) $\Rightarrow$ (FC) and concludes the proof of 4.1. Q.E.D.

If ν is only a quasi-invariant regular Borel measure on Z, then there may not bc any analog of Theorem 4.1; in this case Theorem 3.5 is the best result, i.e., there exist functions $\varphi \in P(Z, \nu)$ which are not moved much by any translation $\varphi \to \delta_x * \varphi$ as x ranges through a prescribed compact set K of translations in G, but there may not be any such function of the elementary form $\varphi_A = [1/\nu(A)] \cdot \chi_A$ for $A \subset Z$.

EXAMPLE. Let $G = Z = \mathbf{Z}$ (the integers) with the usual action $\mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$, and define $\nu(A) = \sum \{e^{|n|} : n \in A\}$ for subsets $A \subset \mathbf{Z}$. Consider the compact set $K = \{2\} \subset G$ and let $A \neq \varnothing$ be any compact set. One can readily perform the computations which show that $(2 + A)\,\Delta A$ contains the two point set $\{\alpha, \beta + 2\}$ where $\alpha = \inf(A) \leqslant \beta = \sup(A)$, and that this implies (independent of α, β):

$$\frac{\nu((2 + A)\,\Delta A)}{\nu(A)} \geqslant e^{-3}.$$

Thus if $\epsilon = e^{-3}$ and $K = \{2\}$, there is no finite set $U \subset Z$ which satisfies the conditions of Theorem 4.1.

5. APPLICATION: A FROBENIUS RECIPROCITY PROPERTY FOR INDUCED REPRESENTATIONS

We presume familiarity with Fell's definition [3], [4] of *weak containment* $L \prec \mathscr{G}$ of a unitary representation L of G in a family $\mathscr{G}$ of such representations, and also with the notion of *weak equivalence* $\mathscr{G} \sim \mathscr{T}$ of two families of unitary representations. Let $\hat{G}$ be the set of (equivalence classes of) irreducible unitary representations of G. Let H be a closed subgroup of G, T a unitary representation of G, and U^T the corresponding induced unitary representation on G (see [9]: we shall discuss various definitions of induced representations below); if S is a representation of G, write $S \mid H$ for its restriction to H (its Hilbert space remaining the same). Fell [5], Section 5, examines the following weak Frobenius reciprocity properties for G.

(WF1) For every $S \in \hat{G}$ and every closed subgroup $H \subset G$, we have $U^{S|H} \succ S$.

(WF2) For every closed subgroup $H \subset G$ and every $T \in \hat{H}$, we have $U^T | H \succ T$.

If G has (WF1) then G must be amenable, for we may take $H = \{e\}$, T any element in $\hat{G}$, and (WF1) then implies that $T \prec U^{T|H}$. But $T \mid H$ is just a direct sum of $\dim(\mathscr{H}_T)$ copies of the one-dimensional identity representation I on $H = \{e\}$, so $T \mid H \prec I$ which $\Rightarrow U^{T|H}$ is weakly contained in U^I; thus $T \prec U^I$ for all $T \in \hat{G}$. But U^I is clearly the left regular representation, and it is a famous result (see [6], Section 3.5) that $\hat{G} \prec U^I \Leftrightarrow G$ is amenable. Thus (WF1) $\Rightarrow$ (amenable). In [5] Fell shows that, for the groups $H = SL(2, \mathbf{C}) \subset G = SL(3, \mathbf{C})$, there is a representation $T \in \hat{H}$ such that $S \mid H$ fails to weakly contain T for any unitary representation S of G; in particular (WF2) fails. Furthermore, $SL(3, \mathbf{C})$ is nonamenable (a semisimple group is amenable $\Leftrightarrow$ it is compact; see [6], Section 3.3), so (WF1) fails.

The considerations of Sections 2 and 3 are just what we need to analyze certain weak containment problems. In particular, we answer a question raised in [5] in the following result.

THEOREM 5.1. *A group G is amenable $\Leftrightarrow$ G satisfies* (WF1).

Before giving the proof for general locally compact G we must reformulate Blattner's definition [9] of induced representations in this context (for separable G one may take Mackey's definition [10] and go directly to the proof). Mackey's construction most clearly displays the connection between amenable action of G on G/H and the weak containment properties of representations induced from H up to G. The following outline shows how Blattner's construction may be cast in the form presented by Mackey. Mackey's proofs do not generalize to the non-separable case, but his construction does.

Let closed subgroup $H \subset G$ be fixed (with $\pi : G \to G/H$ the canonical map) along with a quasi-invariant measure ν on G/H and Haar measures m_G, m_H. Define $\rho(k) = \Delta_H(k)\,\Delta_G(k^{-1})$ for $k \in H$ and let $f \to f^0$ be the map $L(G) \to CB(G)$ which averages $f \in L(G)$ over cosets of $H : f^0(x) = \int f(xk)\, dm_H(k)$. Since $f^0(xk) = f^0(x)$ all $k \in H$, this determines a map $f \to f^{00}$, a linear surjection $L(G) \to L(G/H)$. Finally let β be a Bruhat function on G: $\beta \in CB(G)$ with $\operatorname{supp}(\beta) \cap \pi^{-1}(K)$ compact if K is compact on G/H and $\int \beta(xk)\, dm_H(k) = 1$ all $x \in G$. We remark that if $f, g \in L(G)$ have $f^{00} = g^{00}$, and if φ is any locally m_G-integrable function on G such that $\varphi(xk) = \rho(k)\,\varphi(x)$ all $k \in H$, then $\int f\varphi\, dm_G = \int g\varphi\, dm_G$. Since $f \to f^{00}$ is surjective this allows us to define a Radon measure μ_φ on G/H for any such φ via $\mu_\varphi(f^{00}) = \int f\varphi\, dm_G$.

Now let L be a strongly continuous unitary representation of H in Hilbert space $\mathscr{M}$. The induced representation U^L in Blattner's account is modeled in a space $\mathscr{H}$ of vector valued functions defined as follows. First consider the functions $f : G \to \mathscr{M}$ with properties

(i) f is locally m_G-Bochner measurable
(ii) $f(xk) = \rho(k)^{1/2} L_{k^{-1}} f(x)$ all $k \in H$
(iii) $x \to \|f(x)\|^2$ is locally m_G-integrable,

identifying f, g if $\|f(x) - g(x)\|^2 = 0$ locally m_G—a.e. on G. Now $x \to (f(x) \mid g(x))$ is locally m_G-integrable, from (iii), and the remarks of the last paragraph show that we get a Radon measure $\mu_{f,g}$ on G/H via $\mu_{f,g}(h^{00}) = \int h(x)(f(x) \mid g(x))\, dm_G(x)$ for $h \in L(G)$. We take $\mathscr{H}$ to be those functions f with $\mu_{f,f}$ a *finite* measure. It can be shown that $\langle f, g\rangle \to \mu_{f,g}(G/H)$ makes $\mathscr{H}$ a Hilbert space and we define $U_x{}^L f(t) = f(x^{-1}t)$ to get the strongly continuous *induced unitary representation* $(U^L, \mathscr{H})$.

To reformulate this we let $\nu_x(E) = \nu(x^{-1}E)$ for $x \in G$, where ν is our quasi-invariant measure on G/H, and consider another space $\mathscr{X}$ of functions consisting of those $f : G \to \mathscr{M}$ with

(i) f locally m_G-Bochner measurable
(ii) $f(xk) = L_{k^{-1}} f(x)$ all $k \in K$
(iii) $\int_{G/H} \|f(xH)\|^2 d\nu(xH) < \infty$.

Notice that $x \to \|f(x)\|^2$ is m_G-measurable by (i) and is constant on cosets of H, so (iii) makes sense; likewise for $x \to (f(x) \mid g(x))$. Now $\mathscr{X}$ becomes a Hilbert space if we take

$$(f \mid g) = \int_{G/H} (f(xH) \mid g(xH))\, d\nu(xH).$$

A Borel set $E \subset G/H$ is locally ν-null $\Leftrightarrow$ the (Borel) set $\pi^{-1}(E)$ is locally m_G-null (see [*1*], Section 2, No. 5); thus if $x \in G$ is fixed, the locally ν-integrable Radon-Nikodym derivative $J_x(\xi) = d(\nu_x)/d\nu(\xi)$ gives and m_G-measurable function $J_x \circ \pi$ on G. We assert that we get a strongly continuous unitary representation $(V^L, \mathscr{X})$ which is unitarily equivalent to $(U^L, \mathscr{H})$ by taking $V_x{}^L f(t) = J_x(t)^{1/2} f(x^{-1}t)$ [this is Mackey's construction but his methods are too special to prove the assertion in this general context]. We have only to exhibit an equivariant isomorphism $T : \mathscr{X} \to \mathscr{H}$ which preserves inner products.

If m_G, m_H are fixed, a quasi-invariant measure ν is determined by a locally m_G-integrable function $\varphi > 0$ on G with the property $\varphi(xk) = \rho(k)\, \varphi(x)$ all $k \in H$ via

$$\int_{G/H} h^{00}\, d\nu = \int_G h\varphi\, dm_G \qquad \text{all} \qquad h \in L(G).$$

(see [*1*], Section 2, No.5). We must have $\{x : \varphi(x) = 0\}$ a locally m_G-null set in G. If β is our Bruhat function on G and $f \in \mathscr{X}_0$, the norm dense submanifold of those $f \in \mathscr{X}$ which are continuous on G and have $\pi(\mathrm{supp}(f))$ compact, then $\beta(f \mid f) \in L(G)$ and we have

$$(*) \qquad \int_{G/H} (f \mid g)\, d\nu = \int_G \beta\varphi(f \mid g)\, dm_G \qquad g, f \in \mathscr{X}_0$$

In fact, the right side is well-defined ($\beta(f \mid g)$ is m_G-integrable) and this formula is valid for all $f, g \in \mathscr{X}$.

> [If $f \in \mathscr{X}$ take $\{f_n\} \subset \mathscr{X}_0$ so $f_n \to f$ in $\mathscr{X}$. Then $(f_n \mid f_n)_{\mathscr{X}} \to (f \mid f)_{\mathscr{X}}$ and, taking a subsequence if necessary, we may assume $(f_n(xH) \mid f_n(xH)) \to (f(xH) \mid f(xH))$ ν—a.e. on G/H, $\Rightarrow$ $(f_n(x) \mid f_n(x)) \to (f(x) \mid f(x))$ locally m_G—a.e. in G. Thus if we define $[f \mid f] = \int \beta\varphi(f \mid f)\, dm_G$ we have $[f \mid f] \leqslant \limsup\{\int \beta\varphi(f_n \mid f_n)\, dm_G\} = \limsup\{(f_n \mid f_n)_{\mathscr{X}}\} = (f \mid f)_{\mathscr{X}}$. Thus $[f \mid g] = \int \beta\varphi(f \mid g)\, dm_G$ becomes an inner product dominated by $(\cdot \mid \cdot)$; but $(f \mid g) = [f \mid g]$ for $f, g \in \mathscr{X}_0$, and we conclude $[f \mid g] = (f \mid g)$ all $f, g \in \mathscr{X}$ as required.]

We define $T : \mathscr{X} \to \mathscr{H}$ via $Tf = \varphi^{1/2}f$; it is readily seen that T respects equivalence classes of functions, defines an injective linear map of $\mathscr{X}$ into $\mathscr{H}$ (this requires formula $(*)$ above), and range $(T) = \mathscr{H}$. If $f, g \in \mathscr{X}$ then $(f \mid g) \circ \pi^{-1}$ is supported on a σ-compact set $E \subset G/H$ since $(f \mid g) \circ \pi^{-1}$ is ν-integrable; let $\{\beta_n\} \subset L(G)$ with $0 \leqslant \beta_n(x) \nearrow \beta(x)$ all $x \in \pi^{-1}(E)$. Then $(*)$ gives:

$$
\begin{aligned}
(Tf \mid Tg) &= \mu_{Tf,Tg}(\beta^{00}) \leftarrow \mu_{Tf,Tg}(\beta_n^{00}) \\
&= \int \beta_n(Tf \mid Tg)\, dm_G = \int \beta_n\varphi(f \mid g)\, dm_G \to \int \beta\varphi(f \mid g)\, dm_G \\
&= (f \mid g)_{\mathscr{X}},
\end{aligned}
$$

and only equivariance is left to be proved. But $T^{-1}U_x{}^LT(f)(t) = \varphi^{1/2}(x^{-1}t)\,\varphi^{-1/2}(t)\,f(x^{-1}t)$ and $V_x{}^Lf(t) = (d\nu_x/d\nu)^{1/2}(tH)\cdot f(x^{-1}t)$, so it suffices to show (for fixed $x \in G$) that

$$
\frac{d\nu_x}{d\nu}(tH) = \frac{\varphi(x^{-1}tH)}{\varphi(tH)} \qquad \text{[constant on cosets of } H]
$$

locally ν—a.e. on G/H. But if $f \in L(G/H)$ [say $f = h^{00}$ for $h \in L(G)$] we have

$$
\begin{aligned}
\int_{G/H} f(\xi)\frac{d\nu_x}{d\nu}(\xi)\, d\nu(\xi) &= \int f(x\xi)\, d\nu(\xi) \\
&= \int_G h(xs)\,\varphi(s)\, dm_G(s) = \int_G h(s)\,\varphi(x^{-1}s)\, dm_G(s) \\
&= \int_G h(s)\,\varphi(s)\left[\frac{\varphi(x^{-1}s)}{\varphi(s)}\right] dm_G(s) \\
&= \int_{G/H}\left[\int_H h(sk)\frac{\varphi(x^{-1}sk)}{\varphi(sk)}\, dm_H(k)\right] d\nu(sH) \\
&= \int_{G/H} f(tH)\left[\frac{\varphi(x^{-1}tH)}{\varphi(tH)}\right] d\nu(tH),
\end{aligned}
$$

which suffices to prove our assertion.

Proof of Theorem 5.1. Let H be a fixed closed subgroup, J the identity representation of H, I the identity representation of G. It suffices to show that $U^J \succ I$, for one can show that $U^T \mid H \cong U^J \otimes T$ and $U^J \succ I \Rightarrow U^J \otimes T \succ I \otimes T \cong T$ (see [5], p. 260).[4] Take ν quasi-invariant on G/H. Then G acts amenably on G/H by Corollary 3.2, and by Theorems 3.3, 3.5: if $\epsilon > 0$, K compact subset in G, are given, there exists $\varphi \in L(G/H, \nu)$ with $\| \varphi \|_{1,\nu} = 1$, $\varphi \geqslant 0$ and $\| \delta_x * \varphi - \varphi \|_{1,\nu} < \epsilon$ all $x \in K$. Take $\Phi : G \to \mathbf{C}$ so $\Phi(x) = \varphi^{1/2} \circ \pi(x)$ where $\pi : G \to G/H$. Then $\int_{G/H} | \Phi(xH)|^2 \, d\nu(xH) = \int | \varphi(\xi) | \, d\nu(\xi) = 1$, so $\Phi \in \mathscr{X}(U^J)$ the space of the induced representation U^J, and the positive definite function on G associated with U^J and the vector Φ ($\| \Phi \|_{\mathscr{X}} = 1$) is

$$
\begin{aligned}
(U_x^J \Phi \mid \Phi) &= \int_{G/H} \Phi(x^{-1}tH)\, \overline{\Phi(tH)} \left[\frac{d\nu_x}{d\nu}\right]^{1/2} (tH)\, d\nu(tH) \\
&= \int_{G/H} \varphi^{1/2}(x^{-1}\xi)\, \overline{\varphi^{1/2}(\xi)} \left[\frac{d\nu_x}{d\nu}\right]^{1/2} (\xi)\, d\nu(\xi) \\
&= \int_{G/H} (\delta_x * \varphi)^{1/2}(\xi)\, \overline{\varphi^{1/2}(\xi)}\, d\nu(\xi)
\end{aligned}
$$

(see Lemma 2.2 for last step). Thus

$$
\begin{aligned}
|(U_x^J \Phi \mid \Phi) - 1 | &= \int_{G/H} [(\delta_x * \varphi)^{1/2} - \varphi^{1/2}]\, \overline{\varphi^{1/2}}\, d\nu \\
&\leqslant \left[\int_{G/H} |(\delta_x * \varphi)^{1/2}(\xi) - \varphi^{1/2}(\xi)|^2 \, d\nu(\xi)\right]^{1/2} \left[\int_{G/H} | \varphi(\xi)| \, d\nu(\xi)\right]^{1/2} \\
&\leqslant \left[\int_{G/H} | \delta_x * \varphi(\xi) - \varphi(\xi)| \, d\nu(\xi)\right]^{1/2} \cdot 1 \\
&= [\| \delta_x * \varphi - \varphi \|_{1,\nu}]^{1/2},
\end{aligned}
$$

which is $< \epsilon^{1/2}$ for all $x \in K$ (note that $| \alpha - \beta |^2 \leqslant | \alpha^2 - \beta^2 |$ if $\alpha, \beta \geqslant 0$, to prove the last inequality). Thus $U^J \succ I$. Q.E.D.

Here is another result on weak containment and induced representations which can be established with the tools we have developed. The result was proved, using other methods, by L. Baggett [11] and communicated to me by J. M. G. Fell.

Theorem. *Let G be a separable locally compact group, N a closed normal subgroup which is type I and regularly embedded in G. Assume that G/N is amenable. Then for every irreducible representation $T \in G^\wedge$, there exists an irreducible representation $S \in N^\wedge$ such that $T \prec U^S$.*

[4] The $\cong$ is proved for separable G in [10], Sec. 12; but separability requirements may be dropped since there are only finitely many (H, G)-double cosets in G.

Proof. For this we use freely the machinery due to Mackey for studying induced representations (see [*10*]). Let $T \in G^\wedge$ and consider $T \mid N$ (domain restricted to N); this representation corresponds to a single orbit $\mathcal{O} \subset N^\wedge$ under the action of G defined by $x \cdot S(n) = S(x^{-1}nx)$, all $x \in G$, $n \in N$. If we take any $S \in \mathcal{O}$ and let $K = \{x \in G : x \cdot S = S\}$ then K is a closed subgroup in G, $K \supset N$, and K/N is amenable since it is a closed subgroup of the amenable group G/N (see [*6*], Theorem 2.3.2). Then let $K_S^\wedge = \{R \in K^\wedge : R \mid N$ is a multiple $S\}$. As is well known the induction map $i : K_S^\wedge \to G_{\mathcal{O}}^\wedge = \{T \in G^\wedge : T \mid N$ corresponds to the orbit $\mathcal{O} \subset N\}$ is bijective, where $i(R) = U^R$; thus there exists an $R \in K_S^\wedge$ such that $U^R \cong T$. But $R \mid N = \alpha \cdot S \Rightarrow R \mid N$ and S are *weakly equivalent* in the sense that $R \mid N \succ S$ and $S \succ R \mid N$ (Write $R \mid N \sim S$ for this relation). Inducing $R \mid N$, S up to the subgroup K we get ${}_KU^S \succ {}_KU^{R|N}$ from [*4*], Theorem 4.2.

Now the action $K/N \times K/N \to K/N$ is obviously amenable; since K/N and K give the same operators on K/N, it follows that $K \times K/N \to K/N$ is also an amenable action and thus $R \prec {}_KU^{R|N} \prec {}_KU^S$. Now induce up to G; we have $T \cong U^R \prec U^{R|N} \prec U^S$, as desired, on applying the inducing in stages theorem. Q.E.D.

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